Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Time 1 hour 30 minutes

Paper reference **9FM0/02**

Further Mathematics

Advanced

PAPER 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over ►

P66797A

1/1/1/1/1/

P 6 6 7 9 7 A 0 1 3 2

1. Given that

$$z_1 = 3\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) - \mathrm{i}\sin\left(\frac{\pi}{12}\right)\right)$$

(a) write down the exact value of

(i) $|z_1 z_2|$

(ii) $\arg(z_1 z_2)$

(2)

Given that $w = z_1 z_2$ and that $\arg(w^n) = 0$, where $n \in \mathbb{Z}^+$

(b) determine

(i) the smallest positive value of n

(ii) the corresponding value of $|w^n|$

(3)

(a) Notice given two complex numbers z_1 and z_2 in mod-arg form $(z=r(\cos\theta+i\sin\theta))$

where z_2 has to be converted into

$z_2=\sqrt{2}(\cos(-\pi/12)+i\sin(-\pi/12))$

to make it into an ADDITION seen in the general mod-arg formula

↳ true due to $\cos x$ EVEN FUNCTION

$\cos(x)=\cos(-x)$

+ reflective symmetry over y-axis

↳ and $\sin x$ ODD FUNCTION

$-\sin(x)=\sin(-x)$

now applying z_1 and z_2 to be multiplied - have to multiply moduli and add the arguments : $z_1z_2=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$

(i) $|z_1z_2| = 3\times\sqrt{2}$

$= 3\sqrt{2}$

(ii) $\arg(z_1z_2) = \arg(z_1)+\arg(z_2)$

↳ reading off from the 'θ' in given mod-arg form :

$= \pi/3 + (-\pi/12)$

$= \frac{4\pi}{12} - \frac{\pi}{12}$

$= \frac{\cancel{3}\pi}{\cancel{12}}$

$\boxed{= \pi/4}$

Question 1 continued

(b) using part (a),

$w = 3\sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)$

$\Rightarrow w^n = (3\sqrt{2})^n\left(\cos(\pi/4) + i\sin(\pi/4)\right)^n$

using De Moivre's theorem on the angles

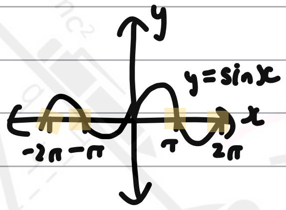
$\hookrightarrow (\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$

can just multiply inner angle by 'n'

$\Rightarrow w^n = (3\sqrt{2})^n\left(\cos(\pi/4 n) + i\sin(\pi/4 n)\right)$

(i) given that $\arg(w^n) = 0$, this suggests that w^n must be completely real, so $\mathrm{Im}(z^n) = 0$

$\therefore \sin(\pi/4 n) = 0$



considering where $\sin = 0$

$\pi/4 n = 0$

$\therefore$ 3 diff. possibilities (ignore -ves as $n \in \mathbb{Z}^+$)

$\pi/4 n = 0$ $\Rightarrow n = 0$ reject as n is an integer

$\pi/4 n = \pi$ $\Rightarrow n = 1$ but then the '4' would also be an 'n' $\therefore$ reject

$\frac{\pi}{4} n = 2\pi$ $\Rightarrow n = 8$ ✓

$\boxed{\therefore n = 8}$

(ii) subbing in $n=8$ into $|w^n|$

$= (3\sqrt{2})^8$

$\boxed{= 104,976}$

(Total for Question 1 is 5 marks)

2.
$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

The matrix **A** represents the linear transformation M.

Prove that, for the linear transformation M, there are no invariant lines.

(5)

(a) METHOD 1 : transforming the line $y = mx + c$

an invariant line is a line of points (call it $y = mx + c$ ∴ as a VECTOR $\begin{pmatrix} x \\ mx+c \end{pmatrix}$)

which under the transformation would be mapped to different points i.e

$\begin{pmatrix} x' \\ mx'+c \end{pmatrix}$ on the SAME straight line

formulating this as an equation (using $Mx = y$)

$$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$$

matrix multiplication "rows into columns" - let product matrix be $\begin{pmatrix} A_{(1,1)} \\ A_{(2,1)} \end{pmatrix}$

...for $A_{(1,1)}$:

$4(x) + ((-2)(mx+c))$

$= 4x - 2mx - 2c$

...for $A_{(2,1)}$:

$5(x) + 3(mx+c)$

$= 5x + 3mx + 3c$

and equating to RHS

$$\begin{pmatrix} 4x-2mx-2c \\ 5x+3mx+3c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$$

$\Rightarrow 4x - 2mx - 2c = x'$ - ①

$5x + 3mx + 3c = mx' + c$ - ②

subbing ① into ②

$5x + 3mx + 3c = m(4x - 2mx - 2c) + c$

expand:

$5x + 3mx + 3c = 4mx - 2m^2x - 2mc + c$

collect x's and c's on either side:

$x(5+m) + 3c = x(4m - 2m^2) + (c - 2mc)$

compare coefficients:

...x:

$5 + 3m = 4m - 2m^2$

$\Rightarrow 2m^2 - m + 5 = 0$

...c:

$(c - 2mc) = 3c$

$2c = -2mc$

$\boxed{\text{but } m \text{ not real}}$

check if real using the determinant: b^2-4ac

$$(-1)^2-4(2)(5)$$
$$=1-4(10)$$
$$=-39<0$$

hence no real solutions for 'm' in $y=mx+c$

$\boxed{\therefore \text{no invariant lines}}$

METHOD 2: transformation of points

an invariant line is a line of points - let the points be $\begin{pmatrix}x\\y\end{pmatrix}$, each of which under the transformation are mapped to another point $\begin{pmatrix}x'\\y'\end{pmatrix}$ on the same line : $y=mx+c$

formulating this as an equation:

$$\begin{pmatrix}4 & -2\\5 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$$

matrix multiplication "rows into columns" a

$$\begin{pmatrix}4x-2y\\5x+3y\end{pmatrix}$$

equate to R.H.S

$$\begin{pmatrix}4x-2y\\5x+3y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$$

into linear equations:

$$4x-2y=x'$$
$$5x+3y=y'$$

now subbing in $y=mx+c$ in both so lie on same straight line

$4x-2(mx+c)=x'$

$\Rightarrow 4x-2mx-2c$

factorise x's and 'c'

$\Rightarrow x(4-2m)-2c$

$5x+3(mx+c)=y'$

$\Rightarrow 5x+3mx+3c=y'$

factorise x's and c's:

$x(5+3m)+3c=y'$

and sub into transformed points: $y'=mx'+c$

$$x(5+3m)+3c=m\left[x(4-2m)-2c\right]+c$$

expand square brackets:

$$x(5+3m)+3c=x(4m-2m^2)+(c-2mc)$$

compare coefficients:

... x:

$$5+3m=4m-2m^2$$
$$\Rightarrow 2m^2-m+5=0$$

check if real using determinant: b^2-4ac

$$(-1)^2-4(2)(5)$$
$$=1-4(10)$$
$$=-39<0$$

... constants:

$$3c=c-2mc$$

but 'm' no real solutions

Question 2 continued

$\therefore$ since m does not $\in \mathbb{R}$ there are
NO invariant lines

(Total for Question 2 is 5 marks)

3. $$f(x) = \arcsin x \qquad -1 \leqslant x \leqslant 1$$

(a) Determine the first two non-zero terms, in ascending powers of x, of the Maclaurin series for $f(x)$, giving each coefficient in its simplest form.

(4)

(b) Substitute $x = \frac{1}{2}$ into the answer to part (a) and hence find an approximate value for π

Give your answer in the form $\frac{p}{q}$ where p and q are integers to be determined.

(2)

(a) know Maclaurin series as an infinitely long polynomial where the coefficients of the powers of 'x' are determined by f(x) and its derivatives all evaluated at 0

$$f(x) = f(0) + f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

(given in formula booklet)

hence need to keep differentiating and evaluating at 0 until we get our non-zero terms

$f(x) = \arcsin x$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$ (from formula booklet)

$= (1-x^2)^{-1/2}$ $\qquad f'(0) = \frac{1}{\sqrt{1-(0)^2}} = 1$

use chain rule (×inner derivative)

$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2} \times -2x$ $\qquad f''(0) = 0(1-0^2)^{-3/2} = 0$

$= x(1-x^2)^{-3/2} \times -2x$ $\qquad$ Or $\frac{0}{(1-0^2)^{3/2}} = 0$

(prep for chain rule)

OR $\frac{x}{(1-x^2)^{3/2}}$ (prep for quotient rule)

$f'''(x)$ - two ways to evaluate this:

WAY 1: product rule - from 'chain rule'

$(1-x^2)^{-3/2} + -\frac{3}{2}x(1-x^2)^{-5/2} \times -2x$ $\qquad f'''(0) = (1)^{-3/2} + 3(0)(0) = 1$

$= (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}$

OR WAY 2: using quotient rule

$u = x \qquad v = (1-x^2)^{3/2}$ $\qquad f'''(0) = \frac{1+0}{1} = 1$

$u' = 1 \qquad v' = \frac{3}{2}(1-x^2)^{1/2} \times -2x$

Question 3 continued

$$= -3x(1-x^2)^{1/2}$$

using rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' + uv'}{v^2}$

$$= \frac{(1-x^2)^{3/2} + 3x^2(1-x^2)^{1/2}}{(1-x^2)^3}$$

$\therefore$ subbing non-zero terms into Maclaurin series formula

$$f(x) = x + \frac{x^3}{3!} + \dots$$

(b) substituting in $x = 1/2$, first into L.H.S

$$\arcsin(1/2) = \sin^{-1}(1/2) = \pi/6$$

then into R.H.S

$$= \left(\tfrac{1}{2}\right) + \frac{\left(\tfrac{1}{2}\right)^3}{3!} + \dots$$

$$= \frac{1}{2} + \frac{1}{48} + \dots$$

$\Rightarrow$ LHS = RHS

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{48}$$

$$\frac{\pi}{6} = \frac{25}{48}$$

but want just π

$$\therefore \pi = \frac{25}{\cancel{48}_{8}} \times \cancel{6}^{1}$$

$$\boxed{= \frac{25}{8}}$$

(Total for Question 3 is 6 marks)

P 6 6 7 9 7 A 0 7 3 2

4. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3 \text{ , } \sum_{r=1}^{n} r^2 \text{ and } \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2 - 1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

METHOD 1: using $(2n-1)^3$

(a) let the cube of a positive odd number be $(2n-1)^3$

$\therefore$ the sum of the cubes of the first n +ve odd numbers (starting from 1 and ending in 'n'), has to be

$$\sum_{r=1}^{n} (2r-1)^3$$

expand inner brackets

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

$$\sum_{r=1}^{n} \left((2r)^3 + 3(2r)^2(-1) + 3(2r)(-1)^2 + (-1)^3\right)$$

$$\Rightarrow \sum_{r=1}^{n} \left(8r^3 - 12r^2 + 6r - 1\right)$$

splitting above into sum of separate terms from CP1.

- sum of n natural numbers = $\sum_{r=1}^{n} 1 = n$
- sum of n constant terms = $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
- sum of squares: $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ → FORMULA BOOKLET
- sum of cubes: $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ → FORMULA BOOKLET

$$\Rightarrow 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$\Rightarrow 8\left(\frac{1}{4}n^2(n+1)^2\right) - 12\left(\frac{1}{6}n(n+1)(2n+1)\right) + 6\left(\frac{1}{2}n(n+1)\right) - n$$

expand brackets

$$\Rightarrow 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$

factorise 'n' out

$$n\left(2n(n+1)^2 - 2(n+1)(2n+1) + 3(n+1) - 1\right)$$

expand inner brackets

Question 4 continued

$\Rightarrow n\left(2n(n^2+2n+1) - 2(2n^2+3n+1) + 3n+3-1\right)$

$\Rightarrow n\left(2n^3 + \cancel{4n^2} + 2n - \cancel{4n^2} - 6n - \cancel{2} + 3n + \cancel{3} - \cancel{1}\right)$

$\Rightarrow n(2n^3 - n)$

factorise another 'n' out:

$\boxed{n^2(2n^2-1)}$

WAY 2 : let the cube of a +ve odd number be $(2n+1)^3$

$\therefore$ the sum of the cubes of the first 'n' +ve odd numbers starting from 1 to 'n' must be

$$\sum_{r=0}^{n-1}(2r+1)^3$$

NOTE: for series to start at 1 need $(2n+1)^3 = (1)^3$

$\Rightarrow 2n+1 = 1$

$\Rightarrow 2n = 0$

$\Rightarrow n = 0$

$\therefore$ for 'n' numbers - go up to 'n-1'

expand inner brackets (used Pascal's triangle)

$$\sum_{r=0}^{n-1}(2r+1)^3 = \sum_{r=1}^{n-1}(8r^3+12r^2+6r+1)$$

splitting above so that can use the four summation formulae

$$\Rightarrow 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r + 1\sum_{r=1}^{n} 1$$

$$= 8\left(\tfrac{1}{4}n^2(n+1)^2\right) - 12\left(\tfrac{1}{6}n(n+1)(2n+1)\right) + 6\left(\tfrac{1}{2}r(r+1)\right) + n)$$

expand brackets

$$= 2n^2(n+1)^2 + 2(n-1)(n)(2n-1) + 3n(n-1) + n)$$

factorise 'n' out:

$$= n\left(2n(n+1)^2 + 2(n-1)(2n-1) + 3(n-1) + 1\right)$$

expand inner brackets

$$= n\left(2n(n^2-2n+1) + 2(2n^2-3n+1) + 3n - 3 + 1\right)$$

$$= n\left(2n^3 - \cancel{4n^2} + 2n + \cancel{4n^2} - 6n + \cancel{2} + 3n - \cancel{3} + \cancel{1}\right)$$

$$= n(2n^3 - n)$$

factorise 'n' out:

$$\boxed{= n^2(2n^2-1)}$$

(b) 3 different methods to find an expression for 10 consecutive odd numbers – all depending on what you first consider the starting term to be

WAY1: evaluate part (a) from $r=n$ to $r=n+9$

$$\sum_{r=n}^{n+9}(2r-1)^3$$

remembering the way to manipulate series when $r\neq 1$

$$\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$$

$$\Rightarrow \sum_{r=1}^{n+9}(2r-1)^3 - \sum_{r=1}^{n-1}(2r-1)^3$$

...subbing in new 'LIMITS' into the result from part(a)

$$(n+9)^2(2(n+9)^2-1) - ((n-1)^2(2(n-1)^2-1) = 99{,}800$$

from question

expand brackets

$$= n^2+18n+81(2(n^2+18n+81)-1) - ((n^2-2n+1)(2(n^2-2n+1)-1)$$
$$= n^2+18n+81(2n^2+36n+161) - ((n^2-2n+1)(2n^2-4n+1)$$
$$= 2n^4+36n^3+161n^2+36n^3+648n^2+2898n+162n^2+2916n+13041$$
$$-(2n^4-4n^3+n^2-4n^3+8n^2-2n+2n^2-4n+1)$$
$$= 2n^4+36n^3+161n^2+36n^3+648n^2+2898n+162n^2+2916n+13041-2n^4+4n^3$$
$$-n^2+4n^3-8n^2+2n-2n^2+4n-1$$
$$= (36+36+4+4)n^3+(161+648+162-1-8-2)n^2+(2898+2916+2+4)n+13041-1$$
$$\Rightarrow 80n^3+960n^2+5820n+13{,}040 = 99{,}800$$
$$\Rightarrow 80n^3+960n^2+5820n-86{,}760=0$$

on CALC-equation solver

$$\Rightarrow n=6$$

$\therefore$ into original expression for cubes of +ve odd numbers

$$(2(6)-1) \boxed{=11}$$

WAY 2: evaluate series from $r=n+1$ to $r=n+10$

$$\sum_{r=n+1}^{r=n+10}(2r-1)^3$$

$$\Rightarrow \sum_{r=1}^{n+10}(2r-1)^3 - \sum_{r=1}^{n+1}(2r-1)^3$$

Subbing limits into part (a)'s expression

$(n+10)^2(2(n+10)^2-1) - n^2(2n^2-1)$

$= n^2+20n+100(2(n^2+20n+199) - 2n^4 + n^2$

expand brackets

$= \cancel{2n^4} + 40n^3 + 199n^2 + 40n^3 + 800n^2 + 3980n + 200n^2 + 4{,}000n + 19{,}900 - \cancel{2n^4} + n^2$

$= (40+40)n^3 + (199+800+200+1)n^2 + (3980+4{,}000)n + 19900 = 99{,}800$

$= 80n^3 + 1200n^2 + 7890n - 79{,}900 = 0$

using CALC and equation solver:

$\Rightarrow n = 5$

Subbing into lower limit

$\Rightarrow 2(5)+1 = \boxed{11}$

WAY 3: evaluate series from $r=n-9$ to $r=n$

$$\sum_{r=n-9}^{n}(2r-1)^3 = \sum_{r=1}^{n}(2r-1)^3 - \sum_{r=1}^{n-10}$$

sub into expression in part (a)

$= (n)^2(2(n)^2-1) - (n-10)^2(2(n-10)^2-1))$

$= n^2(2n^2-1) - [(n^2-20n+100)(2(n^2-20n+100)-1] - ((n^2-20n+100)(2n^2-40n+199))$

expand brackets

$= (2n^2-1) - (2n^4 - 40n^3 + 199n^2 - 40n^3 + 800n^2 - 3980n + 200n^2 + 4{,}000n - 19{,}900$

$= (40+40)n^3 + (-1-199-800-200)n^2 + (3{,}980+4{,}000)n - 19{,}900 = 99{,}800$

$= 80n^3 - 1200n^2 + 7890n - 119{,}700 = 0$

CALC - equation solver

$\Rightarrow n = 15$

Sub into lower limit:

$2(15-9)-1$

$= \boxed{11}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 4 continued

(Total for Question 4 is 9 marks)

5. The curve C has equation

$$y = \arccos\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2$$

(a) Show that C has no stationary points.

(3)

The normal to C, at the point where $x = 1$, crosses the x-axis at the point A and crosses the y-axis at the point B.

Given that O is the origin,

(b) show that the area of the triangle OAB is $\frac{1}{54}\left(p\sqrt{3} + q\pi + r\sqrt{3}\pi^2\right)$ where p, q and r are integers to be determined.

(5)

(a) C having no stationary points suggests that its derivative $\neq 0$; need to differentiate the equation of C

WAY 1: chain rule

using formula for $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$ and chain rule

let $u = \frac{1}{2}x$

$\frac{du}{dx} = \frac{1}{2}$

$\therefore\ y = \arccos(u)$

$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$

subbing in 'u' and multiplying by 'u's' derivative

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= -\frac{1}{\sqrt{1-(\frac{1}{2}x)^2}} \times \frac{1}{2}$

$= -\frac{1}{2\sqrt{1-\frac{1}{4}x^2}}$

which $\neq 0$ as $-1 \neq 0$

$\therefore$ C has no stationary points

WAY 2: using "$\frac{dx}{dy}$"

$y = \cos^{-1}(\frac{1}{2}x)$

taking cos of both sides

$\cos y = \cos(\cos^{-1}(\frac{1}{2}x))$

$\Rightarrow \cos y = \frac{1}{2}x$

$\times 2 \qquad \times 2$

Question 5 continued

$2\cos y = x$

differentiate wrt 'y'

$\frac{dx}{dy} = -2\sin y$

$\therefore \frac{dy}{dx} = \frac{1}{-2\sin y}$

which $\neq 0$ as $1 \neq 0$

$\therefore$ C has no stationary points

(b) first need to find the equation of the normal to C

subbing in $x=1$ into $\frac{dy}{dx}$ (c) from (a)

$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-\frac{1}{4}x^2}}$

$= -\frac{1}{2\sqrt{1-\frac{1}{4}(1)^2}}$

$= -\frac{1}{2\sqrt{3/4}}$

$= \frac{-1}{\frac{\cancel{2}\sqrt{3}}{\cancel{2}}} = -\frac{1}{\sqrt{3}}$

$\therefore$ for gradient perpendicular need -ve reciprocal of above

$\therefore m = \sqrt{3}$

now need to find any coordinate on C - given $x=1$, $\therefore$ Sub into equation of C to get the corresponding 'y' coordinate

$y = \arccos(\frac{1}{2})$

$= \frac{\pi}{3}$

$\therefore$ coordinate: $(1, \frac{\pi}{3})$

↳ knowing 'm' and coordinate of C, can find its equation by subbing into: $y - y_1 = m(x_1 - x_1)$

$\Rightarrow y - \frac{\pi}{3} = \sqrt{3}(x-1)$

expand and make 'y' the subject

$\Rightarrow y = \sqrt{3}x - \sqrt{3} + \frac{\pi}{3}$

Now can find A by making $y=0$ of above

$0 = \sqrt{3}x - \sqrt{3} + \frac{\pi}{3}$

$\Rightarrow \sqrt{3}x = \sqrt{3} - \frac{\pi}{3}$

Question 5 continued

$\div\sqrt{3}$ $\qquad$ $\div\sqrt{3}$

$\Rightarrow x = 1 - \pi/3\sqrt{3}$

$\therefore A = \left(1 - \frac{\pi}{3\sqrt{3}}, 0\right)$

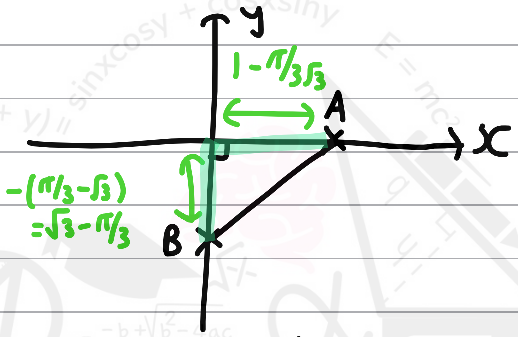
and B by making 'x' $= 0$

$y = \sqrt{3}(0) - \sqrt{3} + \pi/3$

$= -\sqrt{3} + \pi/3$

$\therefore B = (0, \pi/3 - \sqrt{3})$

sketching the triangle OAB



subbing above into 'area of a triangle' formula:

$A = \frac{1}{2}(\sqrt{3} - \pi/3)(1 - \pi/3\sqrt{3})$

expand inner bracket

$= \frac{1}{2}\left(\sqrt{3} - \pi/3 - \pi/3 + \frac{\pi^2}{9\sqrt{3}}\right)$

$\times 1/2$

$= \left(\frac{\sqrt{3}}{2} - \frac{2\pi}{6} + \frac{\sqrt{3}}{54}\pi^2\right)$

$\therefore$ factorise $\frac{1}{54}$ out :

$\frac{1}{54}(27\sqrt{3} - 18\pi + \sqrt{3}\pi^2)$

$\boxed{\Rightarrow p = 27, q = -18, r = 1}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 5 continued

(Total for Question 5 is 8 marks)

6. The curve C has equation

$$r = a(p + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where a and p are positive constants and $p > 2$

There are exactly four points on C where the tangent is perpendicular to the initial line.

(a) Show that the range of possible values for p is

$$2 < p < 4$$

(5)

(b) Sketch the curve with equation

$$r = a(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi \qquad \text{where } a > 0$$

(1)

John digs a hole in his garden in order to make a pond.

The pond has a uniform horizontal cross section that is modelled by the curve with equation

$$r = 20(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$ range

where r is measured in centimetres.

The depth of the pond is 90 centimetres.

Water flows through a hosepipe into the pond at a rate of 50 litres per minute.

Given that the pond is initially empty,

(c) determine how long it will take to completely fill the pond with water using the hosepipe, according to the model. Give your answer to the nearest minute.

(7)

(d) State a limitation of the model.

(1)

(a) we know that for a tangent to be perpendicular to the initial line, $\frac{dx}{d\theta} = 0$

(this is because the gradient of vertical lines is undefined, so from the parametric differentiation of hyperbolic functions:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$ the denominator has to equal 0)

$\therefore$ using parametric definition of (x,y) i.e $(r\cos\theta, r\sin\theta)$

$x = r\cos\theta$

sub in our 'r'

$x = (a(p+2\cos\theta))\cos\theta$

expanding brackets

$x = ap\cos\theta + 2a\cos^2\theta$

now differentiate - 'p' is just a constant
and using chain rule for $\frac{d}{dx}(\cos^2 x)$

$\frac{dx}{d\theta} = -ap\sin\theta + 4a\cos^2\theta(-\sin\theta)$

$= -ap\sin\theta - 4a\cos\theta\sin\theta$

$\times -1$ $\times -1$

$= ap\sin\theta + 4a\cos\theta\sin\theta$

and making it $=0$

$\Rightarrow ap\sin\theta + 4a\cos\theta\sin\theta = 0$

$\div a$ $\div a$

$\Rightarrow p\sin\theta + 4\cos\theta\sin\theta = 0$

factorise $\sin\theta$ out

$\sin\theta(p + 4\cos\theta) = 0$

we know that this linear equation has to produce exactly four solutions

$\sin\theta = 0$ OR $p + 4\cos\theta = 0$

$\theta = \arcsin(0)$ $\Rightarrow 4\cos\theta = -p$

$= 0, (\pi - 0) = \pi$ $\div 4$ $\div 4$

$\Rightarrow \cos\theta = -\frac{p}{4}$

because $\sin\theta = 0$ produces 2 solutions in the range, $\cos\theta = -\frac{p}{4}$ must produce the other 2 solutions

$\therefore$ because $-1 < \cos\theta < 1$ AND p is +ve

means $-\frac{p}{4} > -1$

$\div -4$ $\div -4$ (flip inequality sign)

$\Rightarrow \frac{p}{4} < 1$

$\times 4$ $\times 4$

$\Rightarrow p < 4$

combining this with inequality in the question - $p > 2$

$\Rightarrow \boxed{2 < p < 4}$

(b) equation given in form $a(p + q\cos\theta)$ - remembering rules for sketching polar coordinates:

CASE 1: when $p = q$, get a cardioid (dimple at origin)

CASE 2: $p \geq 2q \rightarrow$ get an oval / 'egg'

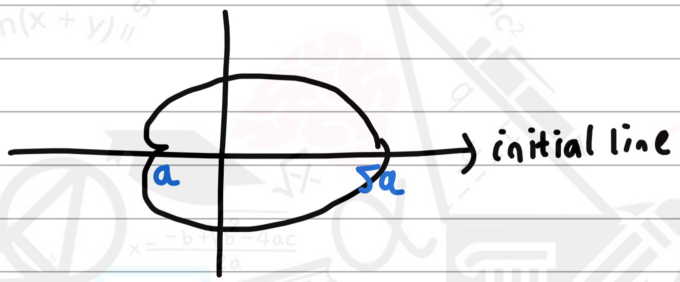
CASE 3: $q < p < 2q \rightarrow$ get a dimple (centre NOT at origin)

now need a table of values to see where dimple is located - go up in increments of $\pi/2$ in the RANGE $0 < \theta < 2\pi$

θ:	0	$\pi/2$	π	$3\pi/2$	2π
r:	$a(3+2)$ $= 5a$	$a(3+0)$ $= 3a$	$a(3+2(-1))$ $= a$	$(3+2(\cos(3\pi/2))$ $= 3a$	$(3+2(1))a$ $= 5a$

dimple

$$\therefore\ r = a(3+2\cos\theta)$$



(c) fiding first area of the uniform horizontal cross-section (polar integration)

WAY 1: limits $(0, 2\pi)$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ((20(3+2\cos\theta))^2 d\theta$$

expanding brackets

$$\Rightarrow 200\int_0^{2\pi} (9+12\cos\theta+4\cos^2\theta)\,d\theta$$

using trig identities: $\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$

$$\Rightarrow 200\int_0^{2\pi} (9+12\cos\theta+2\cos 2\theta+2)\,d\theta$$

$$\Rightarrow 200\int_0^{2\pi} (11+12\cos\theta+2\cos 2\theta)\,d\theta$$

using $\int \cos k\theta\,d\theta = \frac{1}{k}\sin k\theta + c$

$$\Rightarrow 200[11\theta+12\sin\theta+\sin 2\theta]_0^{2\pi}$$

$$\Rightarrow 200\{[11(2\pi)+12\sin(2\pi)+\sin(4\pi)-0]$$

$$\Rightarrow 200(2\pi) = 4400\pi$$

WAY 2: exploiting symmetry (limits $\pi, 0$)

$$2\times\frac{1}{2}\int_0^{\pi} (20(3+2\cos\theta))^2 d\theta$$

$$= 400\int_0^{\pi} 9+12\cos\theta+4\cos^2\theta$$

$$= 400\int_0^{\pi} 9+12\cos\theta+4\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)$$

$$= 400\int_0^{\pi} (11+12\cos\theta+2\cos 2\theta)\,d\theta$$

integrate

$$= 400[11\theta+12\sin\theta+\sin 2\theta]_0^{\pi}$$

$$= 400\{[11(\pi)+12(0)+\sin(0)]-[0]\}$$

$$= 400(11\pi)$$

$$= 4{,}400\pi$$

now to get the VOLUME of the pond, need to do:

$= \text{area of cross section} \times \text{depth}$

$= 4{,}400\pi \times 90$

$= 396{,}000\pi\ cm^3$

but the rate of flow of the water in the hosepipe is given in Litres - so using:

$1\ L = 1{,}000\ cm^3$

$396{,}000\pi\ cm^3 = 396\pi\ L$

$= 396(3.1415..)$

$V = 1244\ L$

∴ the time taken for hosepipe to fill pool must be

$\frac{1244}{50} = 24.88...$

$= \boxed{25\ min}$

(d) possible limitations:

- polar equation may not be accurate
- sides may not be smooth
- hole may not be of uniform depth
- pond may leak (ground absorbs water ∴ calculated volume may be an overestimate)

(Total for Question 6 is 14 marks)

7. **Solutions based entirely on graphical or numerical methods are not acceptable.**

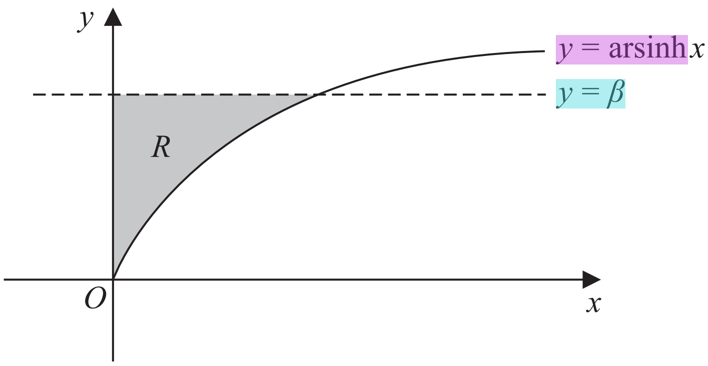


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = \operatorname{arsinh} x \qquad x \geqslant 0$$

and the straight line with equation $y = \beta$

The line and the curve intersect at the point with coordinates (α, β)

Given that $\beta = \frac{1}{2}\ln 3$

(a) show that $\alpha = \dfrac{1}{\sqrt{3}}$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the curve with equation $y = \operatorname{arsinh} x$, the y-axis and the line with equation $y = \beta$

The region R is rotated through 2π radians about the y-axis.

(b) Use calculus to find the exact value of the volume of the solid generated.

(6)

(a) question is asking us to find the point of intersection between $y = \operatorname{arsinh} x$ and $y = \beta = \frac{1}{2}\ln 3$ i.e to solve the equation :

$$\operatorname{arsinh} \alpha = \tfrac{1}{2}\ln 3$$

WAY 1 : using exponential definition of sinhx

$$\sinh^{-1}\alpha = \tfrac{1}{2}\ln 3$$

taking sinh of both sides

$$\sinh\sinh^{-1}\alpha = \sinh\left(\tfrac{1}{2}\ln 3\right)$$

$$\Rightarrow \alpha = \sinh\left(\tfrac{1}{2}\ln 3\right)$$

using $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$ but with $x \to \frac{1}{2}\ln 3$

P 6 6 7 9 7 A 0 2 0 3 2

Question 7 continued

$$\alpha = \frac{e^{\frac{1}{2}\ln 3} - e^{-\frac{1}{2}\ln 3}}{2}$$

Simplifying the logs using power log law:

$$\tfrac{1}{2}\ln 3 = \ln(3)^{1/2} = \ln\sqrt{3}$$

$$-\tfrac{1}{2}\ln 3 = \ln(3)^{-1/2} = \ln\left(\tfrac{1}{\sqrt{3}}\right)$$

$$\alpha = \frac{e^{\ln(\sqrt{3})} - e^{\left(\ln\left(\frac{1}{\sqrt{3}}\right)\right)}}{2}$$

$$= \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{2} = \frac{\frac{(\sqrt{3})(\sqrt{3})}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{2} = \frac{\frac{\cancel{2}}{\sqrt{3}}}{\cancel{2}} = \boxed{1/\sqrt{3}}$$

WAY 2: using formula for arsinh x in formula booklet

$$\text{arsinh}\,\alpha = \tfrac{1}{2}\ln 3$$

$$\text{arsinh}\,x = \ln\left(x + \sqrt{x^2+1}\right)$$

(in formula book)

but with $x \to \alpha$

$$\ln\left\{\alpha + \sqrt{\alpha^2+1}\right\} = \tfrac{1}{2}\ln 3$$

using power log law on R.H.S

$$\Rightarrow \ln\left\{\alpha + \sqrt{\alpha^2+1}\right\} = \ln\sqrt{3}$$

compare the ln bases:

$$\alpha + \sqrt{\alpha^2+1} = \sqrt{3}$$

$$\Rightarrow \sqrt{\alpha^2+1} = \sqrt{3} - \alpha$$

square both sides:

$$\Rightarrow \alpha^2 + 1 = (\sqrt{3} - \alpha)^2$$

expand binomially

$$\Rightarrow \alpha^2 + 1 = \left(3 - 2\sqrt{3}\,\alpha + \alpha^2\right)$$

$$\Rightarrow 2\sqrt{3}\,\alpha = 2 \quad \div 2\sqrt{3}$$

$$\div 2\sqrt{3} \quad \boxed{\alpha = 1/\sqrt{3}}$$

(b) remembering the formula for volumes of revolution about the y-axis

$$V = \pi \int_{\alpha}^{\beta} x^2\,dx$$

and subbing in the equation in the question rearranged:

$$\Rightarrow x = \sinh y$$

P 6 6 7 9 7 A 0 2 1 3 2

Question 7 continued

$$x^2 = \sinh^2 y$$

and $\beta = \frac{1}{2}\ln 3$, $\alpha = 0$

$$\pi \int_0^{\frac{1}{2}\ln 3} \sinh^2 y \, dy$$

...two ways to evaluate this integral:

WAY 1: integration using trig identities

remembering rearranged cos double angle

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

but to get $\sinh^2 x$ – this involves OSBORNE'S RULE i.e because we have the product of two 'sines' - have to negate the standard trig function for $\sin^2 x$

$$\therefore \sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$$

sub into integral

$$\pi \int_0^{\frac{1}{2}\ln 3} \left(\frac{1}{2}\cosh 2y - \frac{1}{2}\right) dy$$

using $\int \cosh kx \, dx = \frac{1}{k}\sinh kx + C$

$$\pi \left[\frac{1}{4}\sinh 2y - \frac{1}{2}y\right]_0^{\frac{1}{2}\ln 3}$$

...sub limits in:

$$\pi \left\{\left[\frac{1}{4}\sinh 2\left(\frac{1}{2}\ln 3\right) - \frac{1}{2}\left(\frac{1}{2}\ln 3\right)\right] - \left[\frac{1}{4}\sinh(0) - \frac{1}{2}(0)\right]\right\}$$

$$\Rightarrow \pi \left\{\frac{1}{4}\sinh(\ln 3) - \frac{1}{4}\ln 3\right\}$$

factorise $\frac{1}{4}$ out

$$\Rightarrow \frac{\pi}{4}\left(\sinh(\ln 3) - \ln(3)\right)$$

using EXPONENTIAL DFN for sinhx

$$\frac{\pi}{4}\left(e^{\ln 3} - e^{-\ln 3} - \ln(3)\right)$$

$$= \pi/4 \left(3 - \frac{1}{3} - \ln(3)\right)$$

$$= \boxed{\pi/4 \left(4/3 - \ln 3\right)}$$

WAY 2: integrating with exponential DFN

rewriting $\sinh^2 y$ as $\left(\frac{1}{2}(e^y - e^{-y})\right)^2$

expand brackets

$$= \frac{1}{4}\left(e^{2y} - 2\cancel{e^y}\cancel{e^{-y}} + e^{-2y}\right)$$

Question 7 continued

$= \frac{1}{4}(e^{2y} - 2 + e^{-2y})$

sub into integral with '1/4' out in front

$\Rightarrow \frac{\pi}{4}\int e^{2y} - 2 + e^{-2y}\, dy$

using exponential integration: $\int e^{ky}dy = \frac{1}{k}e^{ky} + c$

$\Rightarrow \frac{\pi}{4}\left[\frac{1}{2}e^{2y} - 2y - \frac{1}{2}e^{-2y}\right]_0^{\frac{1}{2}\ln(3)}$

$\Rightarrow \frac{\pi}{4}\left[\frac{1}{2}e^{2(\frac{1}{2}\ln 3)} - 2\left(\frac{1}{2}\ln(3)\right) - \frac{1}{2}e^{-\ln(3)}\right)$

$\Rightarrow \pi/4\left[\frac{1}{2}e^{\ln(3)} - \ln(3) - \frac{1}{2}e^{-\ln(3)}\right.$ ← using power log law: $-\frac{1}{2}e^{-\ln 3} \Rightarrow -\frac{1}{2}e^{\ln(1/3)} = -1/6$

$\Rightarrow \pi/4\left(\frac{1}{2}(3) - \frac{1}{6} - \ln(3)\right)$

$\Rightarrow \pi/4\left(\frac{3}{2} - \frac{1}{6} - \ln(3)\right)$

$\Rightarrow \boxed{\pi/4\left(\frac{4}{3} - \ln(3)\right)}$

(Total for Question 7 is 9 marks)

8. (i) The point P is one vertex of a regular pentagon in an Argand diagram.
The centre of the pentagon is at the origin.

Given that P represents the complex number $6 + 6\mathrm{i}$, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$

(5)

(ii) (a) On a single Argand diagram, shade the region, R, that satisfies both

$$|z - 2\mathrm{i}| \leqslant 2 \quad \text{and} \quad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$

(2)

(b) Determine the exact area of R, giving your answer in simplest form.

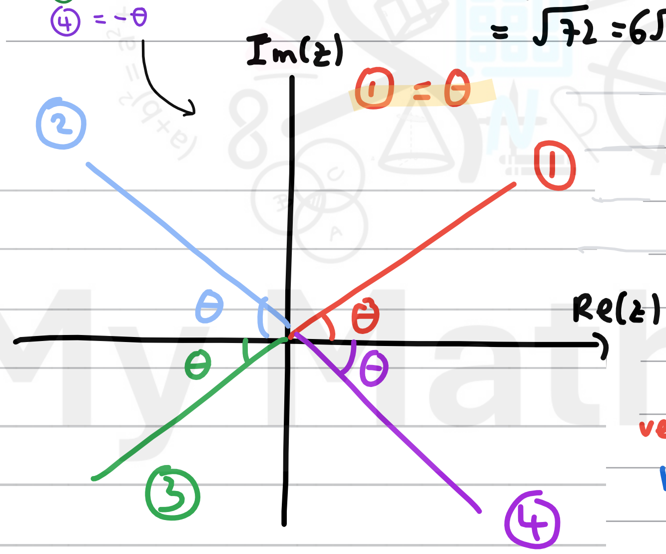
(4)

(a) we can use the properties of complex nth roots to solve geometric problems like this i.e rotate the given complex number vertex about the origin (multiply it by one root of unity 'w' to get other vertices)
·currently P is given in a+bi form - need it in exponential $re^{i\theta}$

① = θ
② = π−θ
③ = −(π−θ)
④ = −θ

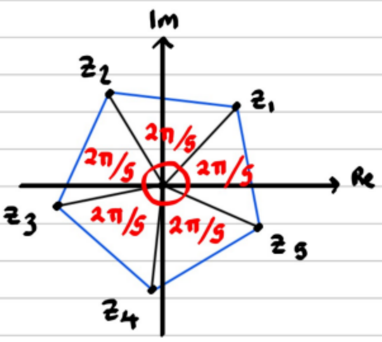
$r = |z| = \sqrt{(a)^2+(b)^2}$
$= \sqrt{36+36}$
$= \sqrt{72} = 6\sqrt{2}$

and $\theta = \arg(z) = \tan^{-1}\left(\frac{6}{6}\right)$
$= \tan^{-1}(1)$
$= \pi/4$
(+ve as in I quadrant - see below)



$\therefore P = 6\sqrt{2}e^{i\pi/4}$

now getting 5th root of unity i.e
$w = e^{i\frac{2\pi}{n}}$
$= e^{i\,2\pi/5}$
↳ representing angle through which vertices are rotated to get the regular pentagon



∴ vertices:
$z_1 = 6\sqrt{2}e^{i\pi/4}$
$z_2 = 6\sqrt{2}e^{i\pi/4} \times e^{i2\pi/5}$
$= 6\sqrt{2}e^{i13\pi/20}$
$z_3 = 6\sqrt{2}e^{i13\pi/20} \times e^{i2\pi/5}$
$= 6\sqrt{2}e^{i21\pi/20}$ (−2π so principal argument form)
$= 6\sqrt{2}e^{i-19\pi/20}$

P 6 6 7 9 7 A 0 2 4 3 2

Question 8 continued

$z_4 = 6\sqrt{2}e^{i(21/20\pi)} \times e^{i2\pi/5}$

$= 6\sqrt{2}e^{i(29/20\pi)}$ (-2π)

$= 6\sqrt{2}e^{i(-11/20\pi)}$

$z_5 = 6\sqrt{2}e^{i(29/20\pi)} \times e^{i2\pi/5}$

$= 6\sqrt{2}e^{(37/20\pi)i}$ (-2π)

$= 6\sqrt{2}e^{-3/20\pi i}$

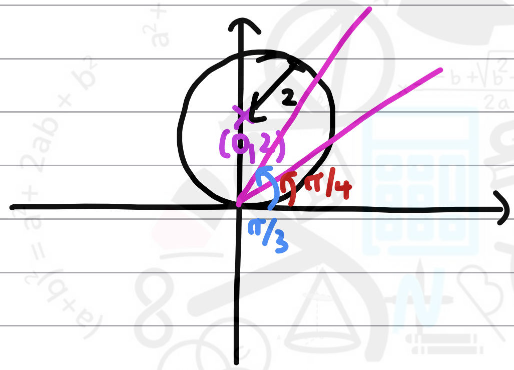
$\therefore$ 5 vertices: $z_2 = 6\sqrt{2}e^{i\,13\pi/20}$

$z_3 = 6\sqrt{2}e^{-19/20\pi\, i}$

$z_4 = 6\sqrt{2}e^{-11\pi/20 i}$

$z_5 = 6\sqrt{2}e^{-3/20\pi i}$

(ii)(a) notice $|z-2i| \leq 2$ is in the form: $|z-z_1| = r$ which represents a circle centre $(0,2)$ rad. 2 (illustrated)

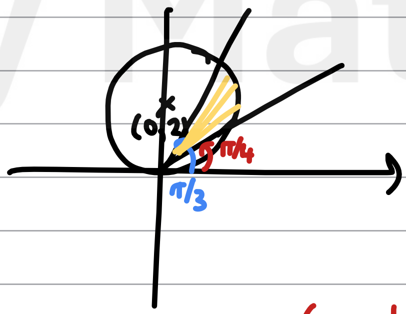


$\frac{1}{4}\pi \leq \arg(z) \leq \pi/3$ is in the form $\arg(z-z_1) = \theta$ which represents a half line extending from but not including $(0,0)$ and making angles from $\pi/3$ to $\frac{1}{4}\pi$ with line parallel to x-axis

(b) METHOD 1: for area of shaded R - consider it radially and use polar integration remembering polar integration formula

$\frac{1}{2}\int r^2\, d\theta$

$\therefore$ need to form a Cartesian equation for the circle (using (ii)(a)) and change it to a polar equation which we can then integrate



circle: $x^2 + (y-2)^2 = 4$

($\rightarrow$ polar; use polar DEN of $x = r\cos\theta$ and $y = r\sin\theta$)

$(r\cos\theta)^2 + (r\sin\theta - 2)^2 = 4$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + 4 = 4$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) - 4r\sin\theta = 0$

using Pythag. identity

$\Rightarrow r^2(1) - 4r\sin\theta = 0$

$\div r$ $\quad \div r$

Question 8 continued

$r = 4\sin\theta$

subbing this into integration of polar equations formula - aswell as $\alpha = \pi/4$ and $\beta = \pi/3$ as limits

$$\frac{1}{2}\int_{\pi/4}^{\pi/3} (4\sin\theta)^2 \, d\theta$$

taking 8 out

$$\frac{8}{2}\int_{\pi/4}^{\pi/3} 2\sin^2\theta \, d\theta$$

evaluate using trig identity: $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ (rearranged cos double angle)

$\therefore 2\sin^2\theta = 1 - \cos 2\theta$

$$4\int_{\pi/4}^{\pi/3} (1 - \cos 2\theta) \, d\theta$$

integrating - use $\int \cos k\theta \, d\theta = \frac{1}{k}\sin k\theta$

$$4\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$4\left\{\left[\pi/3 - \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right] - \left[\pi/4 - \frac{1}{2}\sin(\pi/2)\right]\right\}$$

$$\Rightarrow 4\left[\left(\pi/3 - \sqrt{3}/4\right) - \left(\pi/4 - \frac{1}{2}(1)\right)\right]$$

collecting like terms inside square brackets

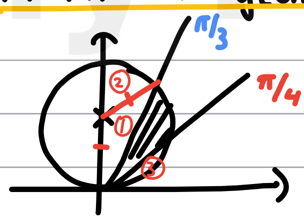
$$4\left[\pi/3 - \pi/4 - \sqrt{3}/4 + 1/2\right]$$

$$4\left[\pi/12 - \sqrt{3}/4 + 1/2\right]$$

expand bracket

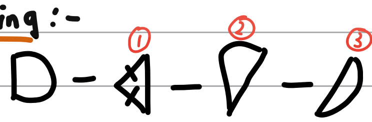
$$\boxed{\Rightarrow \pi/3 - \sqrt{3} + 2}$$

METHOD 2 : geometrical method



WAY 1 : using :-

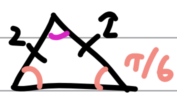
area R =



- area of semicircle

$\frac{1}{2}\pi(2)^2 = 2\pi$

- area 1 - area of isosceles triangle



$\frac{\pi}{2} - \frac{\pi}{3} = \pi/6$

$\therefore$ middle angle

$\pi - \frac{2\pi}{6} = \frac{2\pi}{3}$

$\Rightarrow$ using sine rule $= \frac{1}{2}ab\sin C$

$= \frac{1}{2}(2)^2\sin(2\pi/3)$

$= \cancel{2}(\sqrt{3}/\cancel{2}) = \sqrt{3}$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Question 8 continued

· area ② – area of a sector : $\frac{1}{2}r^2\theta$

$$\frac{1}{2}(2)^2(\pi/3)$$
$$= 2\pi/3$$

· area ③

$$\frac{1}{4}\pi(2)^2 - \frac{1}{2}(2)^2 = \pi - 2$$

$$\therefore R = 2\pi - \left(\sqrt{3} + \frac{2\pi}{3} + (\pi + 2)\right)$$
$$= \left(2\pi - \frac{2\pi}{3}\right) - \sqrt{3} - \pi - 2$$
$$= \frac{4\pi}{3} - \sqrt{3} - (\pi - 2)$$
$$\boxed{= \pi/3 - \sqrt{3} + 2}$$

WAY 2 : area of R =

$- ① - ②$

area of sector:

$$\frac{1}{2}(2)^2(2\pi/3) = 4\pi/3$$

area ① $= \sqrt{3}$ (prev)

area ② $= 2\pi/3$ (prev.)

$$\therefore R = 4\pi/3 - \sqrt{3} - (\pi - 3)$$
$$= \frac{4\pi}{3} - \sqrt{3} - \pi + 3$$
$$\boxed{= \pi/3 - \sqrt{3} + 2}$$

(Total for Question 8 is 11 marks)

9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \dots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \dots$ cannot be purely imaginary, giving a reason for your answer.

(2)

(a) we know from Pure Year 1 that the sum of an infinite series is

$S_\infty = \frac{a}{1-r}$

where 'a' = first term

'r' = common ratio

$|r| < 0$

∴ applying given complex equation to above ($|z| < 0$)

$a = 1$

$r = z$

$\boxed{\therefore S_\infty = \frac{1}{1-z}}$

(b) subbing 'z' into part (a)

$1 + \left(\frac{1}{2}(\cos\theta + i\sin\theta)\right) + \left(\frac{1}{2}(\cos\theta + i\sin\theta)\right)^2 + \left(\frac{1}{2}(\cos\theta + i\sin\theta)\right)^3 + \dots$

expand

$1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos\theta + i\sin\theta) + \frac{1}{8}(\cos\theta + i\sin\theta) + \dots$

and making it equal to the RHS

$= \frac{1}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}$

now need to manipulate this RHS to make it look like the 'show that' in the question

WAY 1: x2 and using exponential form

$\frac{1}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)}$

×2 ×2

$\frac{2}{2 - (\cos\theta + i\sin\theta)}$

using Euler's formula: $\cos\theta + i\sin\theta = e^{i\theta}$

$$\frac{2}{2-e^{i\theta}}$$

now remembering how to manipulate hyperbolic functions where $k-e^{-i\theta}$

i.e multiply by whole denom. with power negated

$$\frac{\times(2-e^{-i\theta})}{\times(2-e^{-i\theta})}$$

$$\Rightarrow \frac{4-2e^{-i\theta}}{4+1-2(e^{i\theta}+e^{-i\theta})}$$

using identity: $e^{i\theta}+e^{-i\theta} = 2\cos\theta$

$$\Rightarrow \frac{4-2e^{-i\theta}}{5-2(2\cos\theta)}$$

$$\Rightarrow \frac{4-2e^{-i\theta}}{5-4\cos\theta}$$

notice how in the "show that"'s LHS only contains sins ∴ hint at

changed LHS to exponential form

$$\mathrm{Im}\left(1+\tfrac{1}{2}e^{i\theta}+\tfrac{1}{4}e^{i\theta}+\dots\right)$$

$$= \mathrm{Im}(LHS) = \mathrm{Im}\left(\frac{4-2e^{-i\theta}}{5-4\cos\theta}\right)$$

$$= \mathrm{Im}\left(\frac{4-2(\cos\theta - i\sin\theta)}{5-4\cos\theta}\right)$$

$$= \boxed{\frac{4+2\sin\theta}{5-4\cos\theta}}$$

WAY 2: using mod-arg form - RHS

$$\frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)}$$

multiply by negated mod-arg

$\times 1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta$

$$= \frac{1\left(1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta\right)}{\left(1-\frac{1}{2}\cos\theta\right)^2 + \left(\frac{1}{2}\sin\theta\right)^2}$$

Question 9 continued

$\therefore \operatorname{Im}\left(1+\frac{1}{2}(\cos\theta+i\sin\theta+\frac{1}{4}(\cos\theta+i\sin\theta)+\ldots\right) = \operatorname{Im}(LHS)$

$= \operatorname{Im}(RHS)$

$$= \frac{\frac{1}{2}\sin\theta}{(1-\frac{1}{2}\cos\theta)^2+(\frac{1}{2}\sin\theta)^2}$$

$$= \frac{\frac{1}{2}\sin\theta}{1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta}$$

$$= \frac{\frac{1}{2}\sin\theta}{1-\cos\theta+\frac{1}{4}(1)} = \frac{\frac{1}{2}\sin\theta \quad \times 4}{\frac{5}{4}-\cos\theta \quad \times 4} \boxed{= \frac{2\sin\theta}{5-4\cos\theta}}$$

using Pythag. identity

WAY 3 : $\times 2$ and negate mod-arg

$$\frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)}$$

$\times 2$ $\quad \frac{2}{2-(\cos\theta+i\sin\theta)}$ $\quad \times 2$ $\quad \times(\cos\theta - i\sin\theta)$ $\quad \times(\cos\theta-i\sin\theta)$

$$= \frac{2(\cos\theta - i\sin\theta)}{(2-\cos\theta)^2+(\sin\theta)^2}$$

$\therefore \operatorname{Im}\left(1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos\theta+i\sin\theta)+\ldots\right) = \operatorname{Im}(LHS)$

$= \operatorname{Im}(RHS)$

$$= \frac{2\sin\theta}{4-4\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= \frac{2\sin\theta}{4+1-4\cos\theta} = \boxed{\frac{2\sin\theta}{5-4\cos\theta}}$$

(ii) WAY 1 : using part (b)(i)

$$S = \frac{4-2\cos\theta}{5-4\cos\theta} + i\left(\frac{2\sin\theta}{5-4\cos\theta}\right)$$

for S to be purely imaginary

$$\frac{4-2\cos\theta}{5-4\cos\theta} = 0$$

$\Rightarrow 4 - 2\cos\theta = 0$

$\Rightarrow 2\cos\theta = 4$

$\div 2 \quad \div 2$

$\cos\theta = 2$

but $-1 < \cos\theta < 1$

$\therefore$ no real solutions
so S cannot be purely imaginary

WAY 2: from (a)

$\frac{1}{1-z} = ki$

$\Rightarrow z = 1 + \frac{1}{k}i$

mod. $|z| > 1$ which is a contradiction

(saw how $|z| < 1$ $\therefore$ convergent)

$\therefore$ not purely imaginary

Question 9 continued

(Total for Question 9 is 8 marks)

TOTAL FOR PAPER IS 75 MARKS